COSMOLOGY AT THE CROSSROADS

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ABSTRACT: Observational tests during the next decade may determine if the evolution of the Universe can be understood from fundamental physical principles, or if special initial conditions, coincidences, and new, untestable physical laws must be invoked. The inflationary model of the Universe is an important example of a predictive cosmological theory based on physical principles. In this talk, we discuss the distinctive fingerprint that inflation leaves on the cosmic microwave background anisotropy. We then suggest a series of five milestone experimental tests of the microwave background which could determine the validity of the inflationary hypothesis within the next decade.

1 Introduction

This talk focuses on how measurements of the cosmic microwave background anisotropy can be used to test the inflationary hypothesis. Compared to the other plenary presentations, the scope may seem rather narrow. This choice has been made intentionally, though, as a means of illustrating the dramatic transformation which cosmology is undergoing and of highlighting why the coming decade is especially critical to the future of this science.

Cosmology is the one of the oldest subjects of human inquiry and, at the same time, one of the newest sciences. Questioning the origin and evolution of the Universe has been characteristic of human endeavor since before recorded history. However, until the 20th century, cosmology lay in the domain of metaphysics, a subject of pure speculation. With little observational evidence to confirm or deny proposals, cosmology could not develop as a true science. In large part, its practitioners were philosophers, religious leaders, teachers and writers, rather than scientists.

The science of cosmology emerged in the 20th Century when it first became possible to probe great distances through the Universe. The great optical telescopes and, later, radio telescopes and satellites provided images and data that could begin to discriminate competing theories. At first, the observational breakthroughs occurred infrequently. Hubble discovered the expansion of the Universe in the 1920's; Penzias and Wilson discovered the cosmic microwave background in the 1960's; and the compelling evidence for primordial nucleosynthesis of the elements was amassed during the 1970's and 1980's.

As the 20th century comes to a close and the new millennium begins, the pace of discovery has accelerated markedly. In the next decade or so, major projects will measure the distribution and velocities of galaxies, dark matter, and radiation at cosmological distances. The new data will completely dwarf all previous observations in quality and quantity. The results will tightly constrain all present theories of the evolution of the Universe and may point to fundamentally new paradigms.

Measurements of the cosmic microwave background anisotropy will be among the most decisive cosmological tests because the microwave background probes the oldest and farthest features of the Universe. Anisotropy measurements will provide a spectrum of precise, quantitative information that, by itself, can confirm or rule out present models of the origin of large

scale structure, reveal the ionization history of the intergalactic medium, and significantly improve the determination of cosmological parameters. More generally, the bounds on human capability to explain the Universe are likely to be decided by what is discovered in the microwave background during the coming decade.

2 At the Crossroads

Cosmology has reached a crossroads which may set its course as a scientific endeavor for the next millennium. By its very nature, the field entails explaining a single series of irreproducible events. Our ability to explore the Universe is physically limited to those regions which are within causal contact. (The causal limit, the maximum distance from which we can receive light or other information, is the Hubble distance $H^{-1} \lesssim 15$ billion light-years.) Given these considerations, it is natural to question whether the evolution of the Universe is completely comprehensible scientifically. Or, more explicitly, which of the following two paths lie ahead of us?

Path I: The basic features of the Universe are explainable as a consequence of symmetry and general physical laws that can be learned and tested near the Earth.

Path II: Some key features of the Universe are largely determined by special initial conditions, extraordinary coincidences and/or physical laws that are untestable locally (*e.g.*, in the most extreme case, accessible only by exploring beyond our causal domain).

At any given time, there may not be complete agreement as to which *Path* we are taking. An observation that seems to suggest special initial conditions (*e.g.*, the flatness of the Universe) may later be explained by new, dynamical concepts (*e.g.*, inflation). This kind of evolution in thinking is common to all sciences. The issue being raised here, though, is whether there is an *ultimate*, fundamental, insuperable limitation to the explanatory and predictive power of cosmology.

Certainly, it is the hope of most cosmologists, at least theorists, that cosmology belongs to *Path I*. We are an ambitious lot, and we aspire to explain

all that is observed. However, nature may not be so kind to human cosmologists. Considering that we are trying to explain a single sequence of events and the range over which we can measure is bounded, it seems plausible that cosmology could ultimately belong to *Path II*. In either case, the exploration of the Universe remains a captivating and important enterprise. But, without doubt, cosmological science is different along the two *Paths*. Along *Path I*, cosmology has the character of physical science, where the field ultimately evolves towards a unified, simple explanation of what is observed. Along *Path II*, cosmology has the character of archaeology or paleontology, where we can classify and quantify phenomena and explain some features, but where many general aspects seem to be accidents of environment or history.

Remarkably, it seems possible that the next decade will determine which *Path* cosmology is to take. Measurements of the microwave background anisotropy and large-scale structure may indicate that the Universe can be explained in terms of a few parameters, known physical laws, and simple initial conditions, all of which suggest *Path I*. Or, we may find that many specially-chosen parameters and complex initial conditions must be invoked, which excludes *Path I* as a possibility.

This review concentrates on testing inflationary cosmology because it is an example of an explanatory model in the sense of *Path I*. Even if inflation is proved wrong, the same tests might indicate if *Path I* might survive under the guise of some other model. The same tests are also relevant for evaluating cold dark matter (CDM) and mixed dark matter (MDM) of large-scale structure formation, which are built upon the conditions created in an inflationary Universe. In addition to testing inflation, the tests might distinguish the CDM vs. MDM possibilities. The discussion is confined to measurements of the microwave background anisotropy because these are precision, quantitative, discriminating tests whose interpretation is least dependent on unverified assumptions.

3 Inflationary Cosmology

The standard big bang model is extraordinarily successful in explaining many features of our Universe: the Hubble expansion, the abundances of light elements, and the cosmic microwave background radiation. However, it does not address some important questions: Why is the Universe so homogeneous?

Why is the Universe spatially flat? Why are there no magnetic monopoles or other remnants from phase transitions that took place early in the Universe? What produced the inhomogeneities in the distribution of matter that seeded the evolution of galaxies? Prior to inflationary theory, the only explanations assumed special initial conditions (suggesting that cosmology is condemned to *Path II*).

Inflationary cosmology [1]–[4] has been proposed as a modification of the standard big bang picture that could explain these mysteries in terms of a well-defined sequence of dynamical processes occurring in the first instants (10^{-35} seconds or so) after the big bang. The central feature is a brief epoch in which the expansion of the Universe accelerates ("inflation"), resulting in an extraordinarily rapid expansion rate. The hyperexpansion of space flattens and smooths the Universe and dilutes the density of monopoles and other remnants to negligible values. Quantum fluctuations produced during the accelerating phase are stretched into a spectrum of energy density perturbations that can seed galaxy formation [5].

Inflation is induced by a change in the equation-of-state. The stretching of a homogeneous and isotropic Universe is described by the Robertson-Walker scale factor, $R(t)$. Inflation or accelerated expansion means that $\ddot{R} > 0$. The time-dependence of $R(t)$ is given by Einstein's equation of motion:

$$\ddot{R} = -\frac{4\pi G}{3}(\rho + 3p)R = -\frac{1}{2}H^2(1+3\gamma)R, \tag{1}$$

where G is Newton's constant, ρ is the energy density, p is the pressure, and H is the Hubble parameter where $H^2 \equiv 8\pi G\rho/3$. The ratio $\gamma = p/\rho$ defines the equation-of-state. Hence, the expansion rate inflates ($\ddot{R} > 0$) if the equation-of-state satisfies $\gamma < -1/3$. Since ρ is always a positive quantity, a large *negative* pressure is required. Any physics which leads to a large negative pressure averaged over cosmological distances (at least a Hubble volume, H^{-3}) can induce inflation.

The standard example is a Universe with energy density dominated by a single, scalar "inflaton" field, ϕ. The equation-of-state is then:

$$\gamma = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}, \tag{2}$$

where V is the effective potential for the inflaton. Here we see that $\gamma < -1/3$ can be achieved if the potential energy density dominates the kinetic energy

density. Inflation continues until ϕ "rolls" to a state of negligible potential energy density. The progress of ϕ can be tracked by solving the Einstein equation, Eq. (1), and the slow-roll equation:

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \tag{3}$$

where $H \equiv d\ \ln R/dt$ is the Hubble parameter and the prime denotes the derivative with respect to ϕ. During inflation, V' must be sufficiently small that the evolution of ϕ is slow, *i.e.*, $\ddot{\phi}$ is negligible. This condition must be maintained long enough for the Universe to have expanded by $N \sim 60$ e-foldings $(R(t_{end})/R(t_{begin}) = \mathrm{e}^{60})$ in order to resolve the cosmological problems of the standard big bang model. As ϕ proceeds towards a steeper part of the potential, ϕ rapidly accelerates and inflation ends. The potential energy, $V(\phi)$, is converted to kinetic energy and, then, is ultimately converted into radiation and matter which reheats the Universe [6, 7].

The most important feature of inflation so far as this talk is concerned is that inflation smoothes out any initial non-uniformity while producing a new spectrum of inhomogeneities [5]. The inflaton and any other light fields all experience quantum de Sitter fluctuations on subatomic scales which inflation stretches to cosmological dimensions. It is convenient to discuss the spectrum of fluctuations in terms of its Fourier components, *i.e.*, a linear combination of plane wave modes. The wavelength of the modes grows as the Universe expands. As the accelerating expansion stretches the wavelength of a given mode beyond the Hubble distance, H_I^{-1} (where H_I is the Hubble parameter during inflation), the amplitude of the quantum de Sitter fluctuations in ϕ (or any other light field) is $\sim H_I/2\pi$. The additional stretching of the wavelength beyond the Hubble length does not change this amplitude since causal physical processes are unable to act over distances greater than the Hubble length. Over the course of inflation, many waves are stretched in this way, ultimately leading to a broad-band spectrum of macroscopic fluctuations with (nearly) the same amplitude.

After inflation ends, the stretching of the Universe decelerates ($\ddot{R} < 0$). The Hubble distance H^{-1} begins to increase at a rate that exceeds the expansion rate, $R(t)$. Hence, even though the waves continue to be stretched, the Hubble distance grows faster, catching up to and ultimately exceeding the wavelengths of some modes. At the point where the Hubble distance equals the wavelength of a given Fourier mode, it is sometimes said that the

"fluctuation re-enters the horizon," referring to the fact that the wavelength was initially less than H_I^{-1} during inflation and has become less than H^{-1} in the post-inflationary epoch. (It would perhaps be more accurate to say "the horizon catches up to the fluctuation.")

The primordial spectrum is determined by the amplitudes of the waves as they re-enter the horizon in the matter- or radiation-dominated epoch. In inflationary models, these amplitudes are precisely the amplitudes as the fluctuations were stretched beyond the horizon during the de Sitter epoch.

The fluctuations of the inflaton, which dominates the energy density of the Universe during inflation, induce a spectrum of energy density perturbations with amplitude:

$$\left.\frac{\delta\rho}{\rho}\right|_{\lambda=H_I^{-1}} \propto \left.\frac{H_I^2}{\dot{\phi}}\right|_{\lambda=H_I^{-1}} \propto \left.\frac{\frac{1}{2}\dot{\phi}^2+V}{\dot{\phi}}\right|_{\lambda=H_I^{-1}}, \tag{4}$$

where H and $\dot{\phi}$ are evaluated as a given wavelength λ is stretched beyond the horizon ($\lambda = H_I^{-1}$) during inflation. Since all microphysical parameters ($V(\phi)$, H, $\dot{\phi}$, etc.) change slowly during inflation compared to the stretching rate ($R(t)$), the ratio in Eq. (4) is nearly constant for all waves. That is, the fluctuations are produced with the (nearly) the same amplitude on average, a nearly scale-invariant (Harrison-Zel'dovich [8]) spectrum of energy-density perturbations.

Inflation also generates similar fluctuations in other light fields. For most fields, these fluctuations are irrelevant because they are insignificant contributors to the total energy density, and the fluctuations leave no distinctive signature. An important exception is quantum fluctuations of massless gravitons, which result in a nearly scale-invariant spectrum of gravitational waves [9]$^-$[11]. Because the gravitational waves are weakly coupled, the spectrum is not erased by reheating or other interactions. Because of their tensor symmetry, their signature on the cosmic microwave background anisotropy is quite different from that of the scalar fluctuations [12, 13]. The predicted gravitational wave amplitude for a mode re-entering the horizon is:

$$|h_k| \;\Big|_{\lambda=H_I^{-1}} \propto \left.\frac{H_I^2}{m_p^2}\right|_{\lambda=H_I^{-1}} \propto \left.\frac{\frac{1}{2}\dot{\phi}^2+V}{m_p^2}\right|_{\lambda=H_I^{-1}}, \tag{5}$$

where m_p is the Planck mass and the expression is to be evaluated as the wavelength is stretched beyond the Hubble length during inflation. As with

the scalar fluctuations, the parameters in the right-hand expression change slowly compared to the stretching so that the gravitational wave spectrum is also nearly scale-invariant.

A critical test for inflation is whether the observed cosmic microwave background anisotropy can be explained in terms of the predicted spectrum of scalar and tensor fluctuations.

4 What Does Inflation Predict?

- *Spatial Flatness*: Inflation flattens the Universe [1], or, more explicitly, exponentially suppresses the spatial curvature contribution to the Hubble expansion relative to the matter and radiation density and relative to any cosmological constant (Λ). Hence, if Ω_{total} is defined as including matter, radiation and vacuum energy density contributions, then inflation predicts that $\Omega_{total} = 1 \pm \epsilon$, where ϵ is exponentially small.

- *Gaussian Primordial Perturbations*: The quantum fluctuations generated in inflation are Gaussian [5]. For a gaussian distribution, the total fluctuation spectrum can be determined from the temperature auto-correlation function (see Section 5).

- *Scale-free Spectrum of Scalar and Tensor Perturbations*: Energy density and gravitational wave fluctuations are generated during inflation with a scale-free spectrum; *i.e.*, a spectrum with no characteristic scale, such as a power-law. The scalar spectrum at time t is conventionally parameterized in terms of its Fourier components by a power-law,

$$\begin{aligned} \left| \left(\delta\rho/\rho|_{\lambda=H_I^{-1}} \right)^2 \right| &\propto k^{n_s-1} \\ \text{or} \quad k^3 \left\langle |\tfrac{\delta\rho}{\rho}(k,t)|^2 \right\rangle &\propto k^{n_s+3}, \end{aligned} \tag{6}$$

  where n_s is the called the *scalar spectral index.* The first expression is in terms of $\delta\rho/\rho$ evaluated at different times, as each mode is stretched beyond the horizon during inflation; the second expression is in terms of $\delta\rho/\rho$ at fixed time. In this convention, $n_s = 1$ corresponds to strict

scale-invariance (Harrison-Zel'dovich [8]). The analogous parameterization for the gravitational wave spectrum is

$$\left|\left(h_k \;\Big|_{\lambda=H_I^{-1}}\right)^2\right| \propto k^{n_t} \quad \text{or} \quad k^3 \left\langle |h_{+,\times}(k,t)|^2 \right\rangle \propto k^{n_t}, \tag{7}$$

where $h_{+,\times}$ are the amplitudes of the tensor metric fluctuations (for two polarizations), n_t is the *tensor spectral index* and $n_t = 0$ corresponds to strict scale-invariance.[1]

In most inflationary models, n_s and n_t actually have "weak" k-dependence. However, microwave background experiments and large-scale structure measurements probe only a narrow range of wavenumbers: Consider a mode with physical wavelength λ today. The physical wavelength at earlier times is $\lambda' = R\lambda$, where we choose the convention that $R = 1$ today. The physical wavelength of the mode at the end of inflation is $\lambda'_{end} = (R_{end}/R_{RH})R_{RH}\lambda$, where R_{RH} is the value of the scale factor after the Universe reheats following inflation. The number of e-folds of inflation between the time that the given mode was stretched beyond the Hubble distance (H_I^{-1}) and the time that inflation ends is

$$\begin{aligned} N(\lambda) &\equiv \ln\left[\tfrac{R_{end}}{R_{RH}} R_{RH}\right] \\ &= 57 + \ln\left(\tfrac{\lambda}{6000\ \mathrm{Mpc}}\right) \\ &\quad + \tfrac{1}{3}\ln\left(\tfrac{V(\phi_{end})^{1/2} T_{RH}}{(10^{14}\,GeV)^3}\right), \end{aligned} \tag{8}$$

where $V(\phi_{end})$ is the potential energy density at the end of inflation, $T_{RH} \leq V(\phi_{end})^{1/4}$ is the temperature at reheating, and 6000 Mpc is the present Hubble distance (for $h = .5$). Microwave background and large scale structure observations span distances between ~ 1 Mpc and ~ 6000 Mpc. These observations cover modes generated during the 10 e-folds between $N(1\ \mathrm{Mpc}) \sim 50$ and $N(6000\ \mathrm{Mpc}) \sim 60$. Over this narrow range, it is an excellent approximation to treat n_s and n_t as

[1] I apologize in behalf of the CMB community for the disgusting convention that defines the indices such that $n_s = 1$ and $n_t = 0$ both correspond to scale-invariant; however, I will maintain the convention in order for readers of this review to be able to comprehend the rest of the literature.

k-independent [14]. In the remaining discussion, n_s and n_t always refer to the values averaged between e-folds 50 and 60.

The total fluctuation spectrum consists of two components, scalar and tensor, each of which is characterized by an amplitude and a spectral index. One convention is to define the amplitudes in terms of the scalar and tensor contributions to the quadrupole moment $C_2^{(S,T)}$ of the CMB temperature autocorrelation function. The scalar and tensor fluctuations are predicted to be statistically independent, so the total quadrupole is the simply the sum of the two contributions. The scalar and tensor quadrupole moments are related to the values of parameters $N_H \equiv N(\sim 6000 \text{ Mpc}) \sim 60$ e-folds before the end of inflation:

$$C_2^S \approx \frac{1}{240\pi^2} \frac{H_I^4}{\dot{\phi}^2}\bigg|_{N_H} \tag{9}$$

and

$$C_2^T \approx 0.073 \frac{H_I^2}{m_p^2}\bigg|_{N_H} . \tag{10}$$

- *Nearly Scale-Invariant Primordial Spectra*: The spectral indices are determined by the equation-of-state, γ, at $50-60$ e-folds before the end of inflation. The derivation of the relations is straightforward and important, so we digress to provide a detailed derivation below. The reader anxious to skip to the answers should proceed to Eqs. (17) and (19) and the discussion below Eq. (19).

  The equation-of-state can be re-expressed in terms of an inflaton field using the relation:

$$\gamma = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{8\pi}{3m_p^2}\left(\frac{\dot{\phi}^2}{H_I^2}\right) - 1, \tag{11}$$

  where $H_I^2 \equiv (8\pi/3m_p^2)[\frac{1}{2}\dot{\phi}^2 + V(\phi)]$. Instead of γ, it is useful in this derivation to introduce a related parameter:

$$\alpha^2 \equiv 24\pi \frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} == 24\pi(1+\gamma) = \left(\frac{8\pi\dot{\phi}}{m_p H_I}\right)^2 . \tag{12}$$

A mode that is stretched so that its wavenumber is $k = H^{-1}$ at time when there are $N(\phi)$ e-foldings remaining has wavenumber $k = H^{-1} \exp N(\phi)$ when inflation ends, where

$$N(\phi) \equiv \int H dt = \int_{\phi}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi. \tag{13}$$

$N(\phi)$ is the number of e-folds that remain before the end of inflation when the inflaton field has value ϕ. Then, we have

$$\frac{d \ln k}{d\phi} = \frac{d\, N(\phi)}{d\phi} = \frac{H}{\dot{\phi}}. \tag{14}$$

The tensor fluctuation amplitude as modes are stretched beyond the Hubble distance inflation is, according to Eqs. (5) and (7), $\propto H_I^2/m_p^2 \propto k^{n_t}$, Hence, the tensor spectral index can be computed according to:

$$\begin{aligned} n_t &= \frac{d \ln (H^2/m_p^2)}{d \ln k} \\ &= \frac{d\phi}{d \ln k} \frac{d \ln (H^2/m_p^2)}{d\phi} \\ &= \frac{\dot{\phi}}{H^3} (H^2)', \end{aligned} \tag{15}$$

where the prime will be used to denote derivatives with respect to ϕ. The inflaton satisfies the equation-of-motion

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \tag{16}$$

but the $\ddot{\phi}$ term is negligibly small during inflation. This also implies that the kinetic energy density is small compared to the potential energy density; or, $(H^2)' \approx 8\pi V'(\phi)/3m_p^2$. Using the slow-roll equation and the expression for $(H^2)'$, we find that:

$$\begin{aligned} n_t &= -\frac{8\pi}{m_p^2} \left(\frac{\dot{\phi}}{H}\right)^2 = -\frac{\alpha^2}{8\pi} \\ &= -3(1+\gamma). \end{aligned} \tag{17}$$

In the end, there is a very simple relation between n_t and the equation-of-state.

The analogous derivation for the scalar spectral index is somewhat more tedious. The amplitude is proportional to $\sim H^4/\dot{\phi}^2 \propto k^{n_s}$. Then, we have

$$\begin{aligned} n_s &= \frac{d\ln(H^4/\dot{\phi}^2)}{d\ln k} \\ &= \frac{d\phi}{d\ln k}\frac{d\ln(H^4/\dot{\phi}^2)}{d\phi} \\ &= 2\frac{\dot{\phi}^2}{H^3}\left[\frac{2HH'}{\dot{\phi}} - \frac{H^2}{\dot{\phi}^2}\dot{\phi}'\right] \\ &= \frac{\dot{\phi}}{H^3}(H^2)' + 2\left(\frac{\dot{\phi}H' - \dot{\phi}'H}{H^2}\right). \end{aligned} \tag{18}$$

The first term, precisely the same as the intermediate expression we obtained for n_t in Eq. (15), is $-3(1+\gamma)$. By the use of Eq. (12) and a bit of algebra, the second term can be expressed as

$$-\frac{m_p}{4\pi}\frac{d\alpha}{d\phi} = \frac{d\ln\alpha^2}{d\ln k} = \frac{d\ln(1+\gamma)}{d\ln k}.$$

Hence, the total expression is

$$n_s = 1 - 3(1+\gamma) + [d\ln(1+\gamma)/d\ln k]; \tag{19}$$

Strict exponential (de Sitter) expansion, $R(t) \propto \exp(H_I t)$, corresponds to $\gamma = -1$, in which limit one obtains precise scale-invariance, $n_s = 1$ and $n_t = 0$, according to Eqs. (19) and (17). However, in any realistic inflation model, the expansion rate must slow down near the end of inflation in order to return to Friedmann-Robertson-Walker expansion. If $R(t)$ is inflating but not exponentially, then $-1/3 > \gamma > -1$, γ' may not be zero, and $n_s \neq 1$ and $n_t < 0$. Inflationary models fall in the range $0.7 \lesssim n_s \lesssim 1.2$ and $0.3 \lesssim n_t \leq 0$; pushing n_s and n_t beyond this range entails exceptional models with special choices of parameters and/or initial conditions [14, 15]. (See comments at the end of this section.)

- *Relations between* ($C_2^{(S)}$, $C_2^{(T)}$, n_t, n_t): COBE and other large-angular scale experiments can determine the total quadrupole moment, $C_2 = C_2^{(S)} + C_2^{(T)}$, placing one constraint on the four parameters that define the inflationary spectrum. The three remaining degrees of freedom are: $r \equiv C_2^{(T)}/C_2^{(S)}$, n_s and n_t. These three parameters are all expressible

Figure 1: The range of the r-n_s plane consistent with generic inflationary models is enclosed by the box. Most models are constrained to lie along the grey diagonal curve; models in which the inflaton encounters an extremum in the inflaton potential near the last 60 e-folds have negligible tensor contribution, $r \approx 0$, along the abscissa inside the box.

in terms of the equation-of-state, γ, and, hence, can be related to one another [12, 13]. The tensor-to-scalar quadrupole ratio, r, is obtained by taking the ratio of Eq. (10) to Eq. (9): $r \approx 173\dot{\phi}^2/(H_I^2 m_p^2)$. Using Eq. (11), we can convert this to:

$$r \equiv C_2^{(T)}/C_2^{(S)} \approx 21(1+\gamma). \tag{20}$$

Comparing the last relation to Eq. (17), we find that

$$r \approx -7n_t. \tag{21}$$

or, we can compare it to Eq. (19) and obtain

$$n_t = n_s - 1 - [d\ln{(1+\gamma)}/d\ln k], \tag{22}$$

and

$$r \approx 7(1-n_s) + [d\ln{(1+\gamma)}/d\ln k] \tag{23}$$

(N.B. r is non-negative, by definition. It is possible to construct models in which the right-hand-side of Eq. (23) is formally negative; this result should be interpreted as indicating negligible tensor contribution, $r \approx 0$. In particular, models with $n_s > 1$, such as some hybrid inflation [16] models, have negligible tensor fluctuations.) These three relations constitute a set of testable signatures of inflation. If observations establish that the primordial spectrum of perturbations is scale-free, observational support for these additional relations would be evidence that the perturbations were generated by inflation.

The predictions are partially illustrated in Fig. 1, which shows a range of parameter-space in the r-n_s plane. Whereas the entire plane describes tensor and scalar perturbations which are scale-free, the range allowed by inflation is confined to the range $0.7 \lesssim n_s \lesssim 1.2$, which is nearly scale-invariant.

Then, Eq. (23) places a constraint on r. Hence, over the entire r-n_s plane, inflationary predictions are confined to a small box. (The boundaries of the box are not precisely defined; one can expand the box 10 per cent or so at the cost of additional fine-tuning of parameters.)

Inflation is falsifiable if observations show that the CMB spectrum lies far outside the box. For example, early reports from COBE analysis suggested that $n_s > 1.5$ [17], which would be inconsistent. These results have since been revised to $n_s \approx 1.0 \pm 0.3$, which is consistent with inflation [18, 19].

Fig. 1 further illustrates that the predictions of inflation do not uniformly cover the box. For most models, $d \ln (1+\gamma)/d \ln k$ is negligible during inflation (which is a way of saying that the equation-of-state changes very slowly during inflation), and, hence,

$$r \approx 7(1 - n_s). \tag{24}$$

These models lie along the grey curve of negative slope shown within the box. A subclass of models has the property that the inflaton encounters an extremum of the inflation potential 60 or so e-folds before the end of inflation. In these models, $d \ln (1 + \gamma)/d \ln k \propto V''(\phi)$ changes significantly during inflation; in particular, $\dot{\phi} \propto V'(\phi)$ shrinks significantly near the extremum, which amplifies the scalar perturbations relative to the tensor (see Eqs. (4) and (5)). Consequently, these models predict that $r \approx 0$, along the abscissa of Fig. 1. Evidence that the CMB spectrum lies in the box but far from the abscissa or the ($r \approx 7(1 - n_s)$) diagonal would be problematic for inflation.

The "generic" predictions of inflation outlined above presume no theoretical prejudice about the "brand" of inflation. They apply to new, chaotic, supersymmetric, extended, hyperextended, hybrid and natural inflation. What determines the prediction is the equation-of-state (*e.g.*, the shape of the inflaton potential $V(\phi)$) during the last 60 e-folds of inflation. Table 1 summarizes the predictions of inflation for some particular forms of the inflaton potential, $V(\phi)$. Since the examples in the Table run the gamut from potentials which are steep to those which are flat, it may be used to estimate the predictions for more general potentials. Conversely, it is possible to reconstruct a section of the inflationary potential over the range of ϕ covered during e-folds 50 to 60 from measurements of the spectral index and the ratio of tensor-to-scalar quadrupole moments [20]. This reconstructed section is extremely narrow since ϕ evolves only a tiny distance down the potential during e-folds

$V(\phi)$	$n_s - 1$	n_t	r
$V_0 \exp(-\frac{c\phi}{m_p})$	$-\frac{c^2}{8\pi}$	$-\frac{c^2}{8\pi}$	$.28c^2$
$A\phi^n$	$-.02 - \frac{n}{100}$	$-\frac{n}{100}$	$.08n$
$V_0 + \lambda\phi^4(\ln\frac{\phi^2}{\sigma^2} - \frac{1}{2})$	$-4 \times 10^{-6}\left(\frac{\sigma}{m_p}\right)^4$	$-.06 - 4 \times 10^{-6}\left(\frac{\sigma}{m_p}\right)^4$	$3 \times 10^{-5}\left(\frac{\sigma}{m_p}\right)^4$
$V_0\left(1 - \frac{\phi^2}{f^2}\right)$	$-\frac{m_p^2}{2\pi f^2}$	$-\frac{\pi m_p^2}{8f^2}\mathrm{e}^{-Nm_p^2/2\pi f^2}$	$2.8\frac{m_p^2}{f^2}\mathrm{e}^{-Nm_p^2/2\pi f^2}$

Table 1: Predictions of inflationary models for some common potentials.

50 to 60. Furthermore, the reconstructed section typically lies far from the false or true vacuum. Hence, the reconstructed section is of limited value in determining the full potential or underlying physics driving inflation.

In the Table, $m_p \equiv 1.2 \times 10^{19}$ GeV is the Planck mass, $N = 60$ is e-folds corresponding to the present horizon (*i.e.*, the natural log of the ratio of the present horizon to the horizon during inflation in comoving coordinates). The first two examples correspond to cases where Eq 24 is a good approximation. The approximation correctly predicts a negligibly small value of r for the third example, but the numerical value is not well-estimated. The predictions for these first three models lie close to the diagonal line in Fig. 1. Eq 24 is a poor approximation for the fourth example, in which the inflaton rolls from the top of a quadratic potential and $d\ln(1+\gamma)/d\ln k$ is large. The prediction for this case is a spectrum with tilt ($n_s < 1$) but insignificant gravitational wave contribution, corresponding to points along the abscissa of Fig. 1.

Exceptional inflationary models can be constructed which violate any or

all of the conditions described above. In fact, because theorists enjoy dwelling on such matters, there are nearly as many papers written on exceptional models as on generic ones. This causes some experimentalists, observers, and non-experts to give these exceptional predictions undue weight. Therefore, it is important to emphasize that these exceptional models, such as those which predict an open or closed Universe or primordial spectra which are not scale-free, are extremely unattractive. First, they require extraordinary fine-tuning beyond what is required to have sufficient inflation and solve the conundra of the big bang model. If one maps out the range of parameter-space which gives sufficient inflation, the exceptional models occupy an exponentially tiny corner. Second, the predictions are not robust. Moving from one point to another within the tiny corner of parameter-space significantly changes the predictions. For example, if one choice of parameters produces a Universe with $\Omega = 0.1$, a slightly different choice increases or decreases the number of e-foldings by one, which results in a change in Ω by a factor of ten. Consequently, there is little or no real predictive power to the exceptional models.

Focusing on the generic tests of inflation is well-motivated for broader reasons: The same tests might determine whether the Universe can be explained on the basis of physical laws testable in the laboratory (*Path I*, as defined in Section 2). The microphysics which we presently understand or can hope to test in the laboratory involves time-scales infinitesimally smaller than the age of the Universe and length-scales infinitesimally smaller than the Hubble length (or the sizes of galaxies). If our Universe is to be comprehended from these physical laws alone without special choices of initial conditions or parameters (*i.e.*, *Path I*), we should not find that there is something special about the present epoch (compared, say, to 10 Hubble times from now or 10 Hubble times earlier) or that there are special features (bumps, dips, etc.) in the primordial spectrum of fluctuations of cosmological wavelength. If we find evidence for new time- or length-scales of cosmological dimensions which cannot be probed in the laboratory, cosmology is thrust into *Path II*. Important features of our Universe must be attributed to initial conditions or physical laws which probably can never be independently tested. (N.B. Testing whether the spectrum is scale-free is less specific than testing inflation. One can imagine finding evidence which supports a flat Universe with scale-free primordial perturbations, but which conflicts with the inflationary relations between r and n_s.)

5 Translating Inflationary Predictions into Precision Tests

The predictions of inflation described in the previous Section can be translated into precise tests of the CMB anisotropy. The implications for the CMB anisotropy can be obtained by numerical integration of the general relativistic Boltzmann, Einstein, and hydrodynamic equations [21]. Included in the dynamical evolution are all the relevant matter-energy components: baryons, photons, dark matter, and massless neutrinos. The temperature anisotropy, $\Delta T/T\,(\mathbf{x}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta,\phi)$, is computed in terms of scalar [22, 23] and tensor [21] multipole components, $a_{\ell m}^{(S)}$ and $a_{\ell m}^{(T)}$, respectively.

The common method of characterizing the CMB fluctuation spectrum is in terms of multipole moments. Suppose that one measures the temperature distribution on the sky, $\Delta T/T\,(\mathbf{x})$. The temperature autocorrelation function (which compares points in the sky separated by angle α) is defined as:

$$\begin{aligned} C(\alpha) &= \left\langle \tfrac{\Delta T}{T}(\mathbf{x}) \tfrac{\Delta T}{T}(\mathbf{x}') \right\rangle \\ &= \tfrac{1}{4\pi} \textstyle\sum_\ell (2\ell+1) C_\ell P_\ell(\cos\alpha), \end{aligned} \tag{25}$$

where $\langle\rangle$ represents an average over the sky and $\mathbf{x}\cdot\mathbf{x}' =\ \cos\alpha$. The coefficients, C_ℓ, are the *multipole moments* (for example, C_2 is the quadrupole, C_3 is the octopole, *etc.*). Roughly speaking, the value of C_ℓ is determined by fluctuations on angular scales $\theta \sim \pi/\ell$. A plot of $\ell(\ell+1)C_\ell$ is referred to as the *power spectrum.* This definition is chosen so that an exactly scale-invariant spectrum, assuming no evolution when the fluctuations pass inside the Hubble horizon, produces a flat power spectrum (*i.e.*, $\ell(\ell+1)C_\ell$ is independent of ℓ). For inflation, the contributions of scalar and tensor fluctuations to the $a_{\ell m}$'s are predicted to be statistically independent. Consequently, the total multipole moment C_ℓ is the sum of the scalar and tensor contributions, $C_\ell^{(S)}$ and $C_\ell^{(T)}$, respectively. The $a_{\ell m}$'s are also predicted to be Gaussian-distributed. The cosmic mean value predicted by inflation is $C_\ell^{(S)} = \left\langle |a_{\ell m}^{(S)}|^2 \right\rangle_E$ and $C_\ell^{(T)} = \left\langle |a_{\ell m}^{(T)}|^2 \right\rangle_E$ where these are averaged over an ensemble of universes $\{E\}$ and over m. The C_ℓ's, which are an average over $2\ell+1$ Gaussian-distributed variables, have a χ^2-distribution.

There is additional valuable information in higher-point temperature correlation functions; *e.g.*, tests for non-gaussianity. However, statistical and

Figure 2: Predicted power spectrum predicted for inflationary cosmology. The spectra are for h=0.5, $\Lambda = 0$, $\Omega_B h^2 = .0125$ and cold dark matter (CDM). The upper (solid) curve has spectral index $n = 1$ (Harrison-Zel'dovich) and pure scalar fluctuations, $r = 0$; the vertical hashmarks represent the one-sigma, full-sky cosmic variance. The lower (dashed) curve has $n = 0.85$ and 50-50 mixture of scalar and tensor quadrupole perturbations, $r = 1$.

systematic errors increase for higher-point correlations; for the short run, the most reliable information will be the angular power spectrum (C_ℓ vs. ℓ).

The predicted spectrum depends not only on the inflationary parameters (r, n_s, n_t) but also on other cosmological parameters. Inflation produces a flat Universe, $\Omega_{total} \approx 1$. In the examples shown in this article, we assume no hot dark matter, $\Omega_{HDM} = 0$, but note that, for angular scales $\gtrsim 10'$, the anisotropy for mixed hot and cold dark matter models with $\Omega_{CDM} + \Omega_{HDM} \approx 1$ is quite similar to the anisotropy if all of the dark matter is cold. For a given value of the Hubble parameter, $H = 100h$ km/sec/Mpc, we impose the nucleosynthesis estimate, $\Omega_B h^2 = 0.0125$, to determine Ω_B. We also satisfy globular cluster and other age bounds [24], and gravitational lens limits [25]: we range from $h \lesssim 0.65$ for $\Omega_\Lambda = 0$ to $h \lesssim 0.88$ for $\Omega_\Lambda \lesssim 0.6$. We also consider a range of reionization scenarios in which the intergalactic medium is fully reionized at some red shift z_R after recombination.

Fig. 2 shows the predicted power spectrum for the central range of inflationary models consistent with the generic predictions outlined in Section 4. Since the value of C_ℓ is determined by fluctuations on angular scales $\theta \sim \pi/\ell$, moving left-to-right in the Figure corresponds to moving from large-angular scales to small-angular scales. Since large-angular scale fluctuations entered the Hubble horizon recently compared to small-angular scale fluctuations, moving left-to-right also corresponds to moving from unevolved primordial fluctuations to fluctuations which have evolved inside the Hubble horizon for a significant time. For these examples, we have chosen $h = 0.5$, $\Omega_\Lambda = 0$, and $\Omega_B = 0.05$.

The predictions of theoretical models, including those for inflation shown in Fig. 2, are expressed in terms of the cosmic mean value of the C_ℓ's averaged over an ensemble of universes or, equivalently, over an ensemble of Hubble horizon-sized patches. In reality, experiments can measure, at best,

over a single Hubble horizon distance. Since the experiments are limited in this way, it is important to know not only the theoretical prediction for the cosmic mean, but also the the theoretical variance about that prediction for experiments confined to a single Hubble horizon distance. This uncertainty, known as the full-sky "cosmic variance," is equal to $C_\ell/\sqrt{2\ell+1}$ for Gaussian-distributed fluctuations, such as those predicted by inflation. Note that the variance decreases with increasing ℓ. Many current small-angular scale experiments cover only a tiny fraction of Hubble horizon-sized patch. If the area fraction of full-sky coverage is A, the theoretical uncertainty scales as $A^{-\frac{1}{2}}$. (In cases where there is much less than full-sky coverage, the theoretical uncertainty is often referred to as "sample variance.")

A more realistic situation is where there are errors due to both sample variance and detector noise [26]. Consider a detection obtained from measurements $(\overline{\Delta T/T})_i \pm \sigma_D$ (σ_D represents detector noise) at $i = 1, \ldots, N_D$ experimental patches sufficiently isolated from each other to be largely uncorrelated. For large N_D, the likelihood function falls by $e^{-\nu^2/2}$ from a maximum at $(\Delta T/T)_{max}$ when

$$\left(\frac{\Delta T}{T}\right)^2 = \left(\frac{\Delta T}{T}\right)^2_{max} \pm \sqrt{\frac{2}{N_D}}\,\nu\,[\left(\frac{\Delta T}{T}\right)^2_{max} + \sigma_D^2]\,. \tag{26}$$

An experimental noise σ_D below 10^{-5} is standard now, and a few times 10^{-6} will soon be achievable; hence, if systematic errors and unwanted signals can be eliminated, the one-sigma ($\nu = 1$) relative uncertainty in $\Delta T/T$ will be from cosmic (or sample) variance alone, $1/\sqrt{2N_D}$, falling below 10% for $N_D > 50$. The hashing in Fig. 2 corresponds to full-sky cosmic variance, roughly equivalent to filling the sky with patches separated by $2\theta_{fwhm}$.

6 The Fingerprint of Inflation

The CMB power-spectrum (Fig. 2) is a fingerprint which is evidence both for inflation and for certain values of cosmological parameters. C_ℓ's for $\ell \lesssim 200$ are dominantly determined by fluctuations outside the Hubble horizon at recombination. These multipoles measure the primordial spectrum of fluctuations. C_ℓ's for $\ell \gtrsim 200$ are sensitive to fluctuations inside the Hubble horizon at recombination. These fluctuations, which had time to evolve prior to last scattering, are sensitive to evolutionary effects which depend on the

Figure 3: Schematic of the power spectrum predicted by a typical inflationary model and the major contributions to it: the Sachs-Wolfe effect (scalar and tensor modes combined), acoustic oscillations of the baryon-photon fluid (density and velocity contributions), and the integrated Sachs-Wolfe effect (significant in models where recombination occurs before matter-domination). [Adapted from Hu & Sugiyama 1994.]

matter density, the expansion rate, and the density, type, and distribution of dark matter. Fig. 3 shows a schematic of the spectrum and the various contributions to it. The distinctive features of the fingerprint are (reading Fig. 2 from left to right):

Plateau at large angular scales ($\ell \lesssim 100$) is due to fluctuations in the gravitational potential on the last-scattering surface (the Sachs-Wolfe effect[27]). Fluctuations in the potential induce red shifts and blue shifts in the CMB photon distribution which create apparent temperature fluctuations. For a precisely scale-invariant ($n_s = 1$) spectrum of scalar fluctuations, the Sachs-Wolfe contribution to C_ℓ is proportional to $1/(\ell(\ell+1))$, and so the contribution to the power spectrum, $\ell(\ell+1)C_\ell/2\pi$ (the ordinate in Fig. 2), is flat. The full computation reveals a slightly upward slope at small ℓ due to other, higher-order contributions, the same effects which are responsible for the Doppler peaks described below. For $n_s < 1$, there is a downward tilt to the Sachs-Wolfe contribution relative to $n_s = 1$ (less power on smaller scales). The $n_s < 1$ curves in Fig. 2 include the tensor contribution predicted by generic inflation models, as expressed by Eq. (23). From observations at small ℓ only, it is difficult to distinguish the tensor contribution because the spectral slope due to the Sachs-Wolfe effect is nearly the same for tensor and scalar contributions for $\ell > 3$.[2] The slope is not very sensitive to the value of h, Ω_B, Ω_Λ or other cosmological parameters.

First Doppler Peak at $\ell \approx 200$ probes wavelengths smaller than the horizon at last-scattering ($\lesssim 1^\circ$). Gravitational wave perturbations begin to oscillate and red shift away once their wavelength falls within the horizon. Hence, for $\ell \gtrsim 200$, the tensor perturbations do not contribute significantly, even if they were dominant over the scalar fluctuations at larger angular scales

[2] There is an intriguing, notable difference in the ratio of the mean quadrupole-to-octopole moment for the models with tensor contribution. However, the effect is not a useful discriminant because there is a large cosmic variance for the small-ℓ multipoles.

($\ell \ll 200$).

The prominent features, known as the Doppler peaks, are due to scalar fluctuations. Fig. 3 illustrates both the acoustic density and acoustic velocity contributions. The peaks are the remnant of adiabatic oscillations in the baryon-photon fluid density. The oscillations in a given mode begin when the wavelength falls below the Jeans length (*i.e.*, pressure dominates over gravity).[3] The Jean's length near recombination is roughly $2\pi c_s H^{-1}$, where $c_s \approx 1/\sqrt{3}$ is the sound speed.

The value of ℓ at the maximum [28] of the first Doppler peak is $\ell_{peak} \approx 220/\sqrt{\Omega_{total}}$. Since the location is insensitive to the value of h or Ω_B, measuring ℓ_{peak} is a novel means of measuring Ω_{total}. The height of the Doppler peak depends on the primordial spectral amplitude, the scalar spectral index (n_s), the tensor-to-scalar quadrupole ratio (r), the expansion rate, and the pressure [26]. If the spectrum is fixed at large angular scales by COBE DMR, smaller values of n_s imply decreasing primordial amplitudes on smaller angular scales and, consequently, a smaller Doppler peak. Gravitational waves add to the plateau at $\ell < 200$, but their contribution to the Doppler peak is red shifted to insignificant values. Consequently, increasing r decreases the height of the Doppler peak relative to the plateau. Increasing the expansion rate (h) pushes back matter-radiation equality relative to recombination, thereby increasing the adiabatic growth of perturbations. Photons escaping from the deeper gravitational potential are red shifted more. Hence, increasing h suppresses the Doppler peak. Increasing the pressure (by decreasing $\Omega_B h^2$) also decreases the anisotropy since the fluctuations stop growing once pressure dominates the gravitational infall. (According to the last two remarks, increasing h produces opposite effects. For $\Omega \lesssim 0.1$, the net effect is that increasing h decreases the anisotropy [29].) The height of the first Doppler peak is relatively insensitive to whether the dark matter is cold or a mixture of hot and cold dark matter.

Second and Subsequent Doppler peaks are due to modes that have undergone further adiabatic oscillations. The peaks are roughly periodically-spaced. The deviation from periodicity is due to time-variation in c_s. The amplitudes decrease as n_s decreases and r increases.

[3] The term, "Doppler peak," is a misnomer since, with standard recombination, the electrons and photons oscillate together and there is little difference in their velocities. The Doppler effect does not become significant until $\ell \gtrsim 1000$.

Anisotropies are caused by inhomogeneities in the baryon-photon fluid density and velocity. The acoustic density and velocity oscillations are 180 degrees out-of-phase with one another, as shown in Fig. 3. The acoustic density contribution is larger. The Doppler peak maxima and minima correspond to maxima and minima of the acoustic density oscillations; the minima do not extend to zero because they are filled in by the maxima in the acoustic velocity contribution [23, 29, 30]. The first and other odd-numbered peaks correspond to compressions and the even-numbered peaks correspond to rarefactions. Gravity tends to enhance the compressions and suppress the rarefactions. The effect is especially noticeable at low pressure (high $\Omega_B h^2$), for which the even-numbered peaks are greatly suppressed or absent altogether. The peaks are also sensitive to whether the dark matter is cold or a mixture of hot and cold.

Damping at $\ell \gtrsim 1000$: CMB fluctuations are suppressed by photon diffusion (Silk Damping [31]). The baryons and photons are imperfectly coupled. The photons tend to diffuse out of the fluctuations and smooth their distribution. Through their collisions with the baryons, the baryons distribution is smoothed as well, thereby suppressing the anisotropy. A second damping effect is due to the destructive interference of modes with wavelengths smaller than the thickness of the last-scattering surface [22, 23]. Doppler peaks from five or so adiabatic oscillations can be distinguished before the damping overwhelms.

7 Five Milestones for Testing Inflation

The fingerprint imprinted by inflation on the CMB anisotropy suggests a series of five milestone tests. Below, the proposed tests are compared with current experimental results. Included are [32]–[42]: COBE DMR (COsmic Background Explorer Differential Microwave Radiometer), FIRS (Far InfraRed Survey), TENerife, SP91 and SP94 (South Pole 1991 and 1994), BP (Big Plate), PYTHON, ARGO, MAX (Millimeter Anisotropy Experiment), MSAM (Medium Scale Anisotropy Experiment), White Dish, and OVRO7 (Owens Valley Radio Observatory, 7 degree experiment).

Milestone 1: Observation of Large-Scale Fluctuations with $\Delta T/T \approx 10^{-5}$

The nearly scale-invariant spectrum of fluctuations generated by inflation

includes modes whose wavelength is much greater than the Hubble horizon at recombination ($\gg 1^\circ$). If inflation is responsible for the formation of large-scale structure, the magnitude of the perturbations should be at the level of $\Delta T/T \approx 10^{-5}$. If $\Delta T/T$ were a factor of five or more smaller, the amplitude would be too small to account for galaxy formation. A value of $\Delta T/T$ a factor of five or more greater would lead to unacceptable clumping of large-scale structure.

In starting a long journey, it is encouraging to begin from a point where the first milestone has already been passed. In this case, COBE DMR [32], with some corroboration from the FIRS [33] and Tenerife [34] observations, finds $\Delta T/T = 1.1 \pm 0.1 \times 10^{-5}$ scales (two-year result for 53 GHz scan smoothed over 10°) [17], just within the range consistent with inflation and dark matter models. Other models, such as cosmic defects and isocurvature baryon (PIB) models, are also consistent with these observations.

Milestone 2: Observation of Scale-free and Nearly Scale-Invariant Spectrum at Intermediate Angular Scales ($20^\circ \gtrsim \theta \gtrsim 1^\circ$)

Inflation predicts a primordial spectrum that is scale-free and nearly scale-invariant. Fig. 4 shows the predictions for the central range of parameters consistent with inflation compared with current measurements of the CMB anisotropy. The curves have been normalized to the COBE DMR two-year result.

A notable effect is that the spectrum for the $n_s = 1$, scale-invariant (Harrison-Zel'dovich) spectrum is not precisely flat. The slight, upward tilt is due to the small contributions of short wavelength ($\lesssim 1^\circ$) modes, which include effects other than simple Sachs-Wolfe. The same contributions become dominant at $\ell \geq 100$ and are responsible for the Doppler peaks addressed by Milestones 3 and 4. Consequently, the *apparent* spectral index — the slope of C_ℓ vs. ℓ determined directly from the CMB anisotropy — differs from the *primordial* spectral index which generated the spectrum. For example, the upper curve has been computed for a primordial index of $n_s = 1$, but that apparent index (the upward tilt) corresponds to $n_s^{app} = 1.15$, a 15 per cent correction. An important feature of the added contributions is that, over the range $\ell \lesssim 100$, they are relatively insensitive to the value of cosmological parameters (h, Ω_B, and Ω_Λ). Fig. 6 illustrates the range of predictions for fixed spectral index ($n_s = 1$) when all other parameters are varied by the maximal amounts consistent with astrophysical observations. The spectra

form a narrow sheath around the original spectrum, clearly separated from the prediction for $n_s = 0.85$. Hence, the spectrum of C_ℓ's for $\ell \lesssim 100$ can test whether the spectrum is scale-free and measure the spectral index without any additional assumptions about other cosmological parameters.

Fig. 4 also shows current detections. The error bars represent the one-sigma limits. (The method of flat band power estimation [44, 45], an important tool for converting experimental results into model-independent bounds on the C_ℓ's, is discussed in the Appendix.) The theoretical curves are normalized to match COBE DMR (two-year). The experiments on this figure illustrate the diversity in CMB experiment. COBE DMR is aboard a space-borne platform; FIRS is a high-altitude balloon experiment; Tenerife is mounted on a mountain in the Canary Islands; South Pole is in Antarctica, of course; and Big Plate is set on the ground in Saskatoon.

At this point, experiments are consistent with a scale-free form for the power spectrum with spectral-index near $n_s = 1$ and $n_t = 0$, but the error bars are large. After four years of data, COBE DMR may be able to reduce its error by two. Much more dramatic improvements are expected for the other experiments. The chief limitation is small sample area (and, hence, overwhelming sample variance) which should be overcome with continued measurements.

By itself, COBE DMR is not a powerful discriminant among theoretical models. For example, Fig. 7 shows the cross-correlation between the 53 GHz and 90 GHz frequency channels. The cross-correlation for a full-sky map is $K(\alpha) \equiv \langle \Delta T_{53}(\mathbf{x}) \Delta T_{90}(\mathbf{x}') \rangle$ where $\mathbf{x} \cdot \mathbf{x}' = \cos\, \alpha$; the coefficients of the multipole expansion of $K(\alpha)$ analogous to Eq. (25) are K_ℓ. If COBE DMR provided a full-sky map and there was no detector noise or foreground contamination, the K_ℓ should equal C_ℓ. In reality, COBE DMR analysis is based on a full-sky from which the galaxy has been cut. Plotted in Fig. 7 are the coefficients of orthonormal moments on the cut sky, using methods developed by Gorski [18]. The fact that there is noise in the COBE experiment accounts for the anticorrelation at large ℓ. At $\ell \lesssim 20$, though, the two channels are highly correlated (open circles) and there is strong evidence for the CMB signal. Superimposed on the plot are predictions for the cross-correlation for diverse models. The dotted lines correspond to one-sigma cosmic variance. The figure shows that the models cannot be distinguished to better than cosmic variance.

Fig. 4 and Fig. 7 illustrate important lessons concerning COBE DMR:

Figure 4: Blow-up of Fig. 2 showing the power spectrum for intermediate angular scales, $2 < \ell < 100$. Note that the upper curve, which corresponds to $n_s = 1$ (scale-invariant), has an upward tilt corresponding to an apparent spectral index of ~ 1.15 at large angular scales. Superimposed are the experimental flat band power detections with one-sigma error bars (see Appendix) for: (a) COBE; (b) FIRS; (c) Tenerife; (d) South Pole 1991; (e) South Pole 1994; (f) Big Plate 1993-4; and (g) PYTHON.

Figure 5: CMB spectra through the first Doppler peak. The solid and dashed black curves correspond to the inflationary/CDM predictions for spectral index ($r = 0$, $n_s = 1$) (Harrison-Zel'dovich) and ($r = 1$, $n = 0.85$), respectively. The dot-dashed curve is the prediction for cosmic texture models. For the data, error bars represent one-sigma. The experiments correspond to: (a) COBE; (b) FIRS; (c) Tenerife; (d) South Pole 1991; (e) South Pole 1994; (f) Big Plate 1993-4; (g) PYTHON; (h) ARGO; (i) MSAM (2-beam) - upper point uses entire data set, the lower point has unidentified point sources removed; (j) MAX3 (GUM region); (k) MAX3 (μPegasus region) showing here unidentified residual after dust subtraction; and (l) MSAM (3-beam) - the upper point uses the entire data set, lower point has unidentified point sources removed.

Figure 6: Effect of varying cosmological parameters on the power spectrum over intermediate ($2 < \ell \leq 100$) scales. The solid curve is the baseline spectrum, $n_s = 1$ (Harrison-Zel'dovich), $r = 0$, h=0.5, $\Omega_\Lambda = 0$ and $\Omega_B h^2 = .0125$. The enveloping dashed curves also have $n_s = 1$, but other cosmological parameters are varied within their astrophysical bounds. Dashed curves from top to bottom: (1) Ω_Λ increased to 0.6; (2) h decreased to 0.4; (3) $\Omega_B h^2$ induced to 0.025; (4) $\Omega_B h^2$ reduced to 0.005; (5) h increased to 0.75. Curves (2) and (3) are difficult to distinguish because they overlap considerably. Note how this whole family of curves is well-separated from the lower dot-dashed curve which corresponds to $n = .85$ and $r = 1$.

Figure 7: Cross-correlation between 53 GHz and 90 GHz frequency channels in the COBE DMR 2-year map compared with three theoretical models: (a) flat CDM universe (solid curve); (b) Λ-dominated flat universe with $\Omega_\Lambda = 0.8$ (upper dashed curve); (c) open baryon-dominated universe with $\Omega = 0.03$ (lower dashed curve). The cross-correlation has been expanded into orthogonal functions on the cut sky labeled by the index ℓ. (Courtesy of K. Gorski.)

First, the initial emphasis on extracting the COBE quadrupole moment and determining the spectral index has perhaps been misplaced. The quadrupole is difficult to extract empirically because of the galactic background. It has limited use theoretically because of the large cosmic variance for this multipole (as shown in Fig. 7). Most important, the quadrupole contains no information that cannot be obtained by measuring the higher multipole moments, which can be fixed empirically and theoretically to greater accuracy. The second lesson is that measuring the spectral index using COBE DMR alone is limited by the fact that it is based on the detection of the first 20-30 multipole moments only. The range of ℓ is rather short for determining accurately the tilt in the power spectrum with increasing ℓ. Fig. 4 suggests that experiments at somewhat larger ℓ ($\gtrsim 50$), when combined with COBE DMR, can produce a more accurate measure of the spectral index because of the larger lever-arm in ℓ. There is the additional advantage that the larger ℓ multipole moments have smaller cosmic variance.

Milestone 3: Observations of the First Doppler Peak

The existence of the first Doppler peak is a generic feature of inflationary models (and CDM or HDM models of large-scale structure formation) assuming standard recombination or reionization at $z \lesssim 100$. The position of the Doppler peak, at $\ell \approx 220/\sqrt{\Omega_{total}}$, tests whether the Universe is spatially flat, as inflation predicts [28]–[30]. The height of the first Doppler peak depends sensitively on h, Ω_B, and Ω_Λ, and on the reionization history. Discriminating the various effects is difficult since rather different choices of cosmological parameters can lead to virtually indistinguishable CMB anisotropy spectra. We refer to this problem as *cosmic confusion*. Its implications are discussed in the next section.

Fig. 5 illustrates the theoretical predictions and the present experimental

limits. Error bars represent one-sigma bounds. It is difficult to draw any firm conclusions from this range of observational results. However, it is interesting to note that a sequence of new MAX results re-examining the GUM (Gamma Ursa Major) and two other regions are consistent with the large amplitude found by the earlier MAX3 GUM experiment (see Fig. 8). Present results suggest a rather large Doppler peak.

Observation of the first Doppler peak is an especially important discriminant between cosmic defect (strings, textures, etc.) and inflationary/CDM models for large-scale structure formation. Fig. 5 also illustrates the predictions for cosmic texture models [46]. The predictions for other cosmic defect models are similar. The cosmic texture predictions shown here assume reionization at $z \gtrsim 200$, which accounts partially for the suppression of fluctuations just where inflation predicts a Doppler peak. However, even if no reionization is assumed, cosmic defect models normalized to COBE DMR predict smaller amplitude fluctuations at 1° scales than inflationary/CDM models (which is related to why they require high bias parameters) [47]. For both reasons, cosmic defect models generally predict a signal considerably less than the Doppler peak predicted for $n_s = 1$ inflationary models, although precise calculations of the anisotropy are not yet available for cosmic defect models with standard recombination.

Milestone 4: Observations of Second and Higher Order Doppler Peaks

If the second Doppler peak can be resolved to near cosmic variance uncertainty, additional constraints on $\Omega_B h^2$ can be extracted. For the standard value $\Omega_B h^2 = 0.0125$, there is a sizable second peak, but this peak is increasingly suppressed as $\Omega_B h^2$ increases (see Fig. 9 and discussion in Section 6). Also, the second and higher order Doppler peaks probe sufficiently small scales to test whether the dark matter is cold or a mixture of hot and cold.

An increasing challenge for observers as the range proceeds to smaller angular scales is foreground subtraction from sources and from the Sunyaev-Zel'dovich effect. Fig. 9 illustrates the few experimental results that presently span this range.

Milestone 5: Observations of Damping

If experiments successfully trace the predicted power spectrum through the second Doppler peak, there is already overwhelming evidence in favor of inflation and dark matter models of large-scale structure formation. Ob-

Figure 8: Blow-up of the power spectrum over a narrow range of ℓ near the left (low ℓ) slope of the first Doppler peak showing recent MAX4 results from the ι-Draconis, GUM (Gamma Ursa Major), and σ-Hercules regions. The two theoretical curves are ($r = 0$, $n_s = 1$, solid) and ($r = 1$, $n_s = 0.085$, dashed) inflationary/CDM predictions, respectively. The dot-dashed curve shows the prediction for cosmic textures (with reionization). The data are: (a) MAX4 GUM 6 cm^{-1} (triangle) and 9 cm^{-1} (diamond) channels; (b) MAX4 ι-Draconis 6 cm^{-1} (triangle) and 9 cm^{-1} (diamond) channels; (c) MAX4 σ-Hercules 6 cm^{-1} (triangle) and 9 cm^{-1} (diamond) channels; (d) MAX4 average (open circles) over all channels for (left-to-right) GUM, ι-Draconis and σ-Hercules; (e) MAX4 GUM 3.5 cm^{-1} for (left-to-right) GUM, ι-Draconis and σ-Hercules; (f) MSAM2 (2-beam) - the upper point is based on the entire data set, the lower point has unidentified point sources removed; (g) MAX3 GUM; (h) MAX3 μ Pegasus (showing here unidentified residual after dust subtraction).

Figure 9: Power-spectrum beginning from $\ell = 50$ spanning all Doppler peaks and the Silk damping region with experimental detections superimposed. Error bars represent one-sigma detections and triangles represent 95% confidence upper limits. The two curves both correspond to pure scalar ($r = 0$), $n_s = 1$ spectra with $h = 0.5$. The solid curve corresponds to the standard value $\Omega_B h^2 = 0.0125$ and the dot-dashed curve corresponds to $\Omega_B h^2 = 0.025$. Note that the second Doppler peak is suppressed relative to the first and second as $\Omega_B h^2$ increases. The experiments correspond to: (a) South Pole 1991; (b) South Pole 1994; (c) Big Plate 1993-4; (d) PYTHON; (e) ARGO; (f) MSAM (2-beam) - the upper point uses the entire data set, the lower point has unidentified point sources removed; (g) MAX3 (GUM region); (h) MAX3 (μPegasus region) showing here unidentified residual after dust subtraction; and (i) MSAM (3-beam) - the upper point uses the entire data set, the lower point has unidentified point sources removed;. (j) White Dish; and (k) OVRO7.

servation of damping at very small angular scales ($\lesssim 5'$) due to photon diffusion (Silk damping [31]) and interference through the thickness of the last-scattering surface is corroboration of more subtle effects on the cosmic microwave background [22]. If there was significant reionization (which can already be determined by experiments at larger angular scales), then experiments probing this range might find evidence for secondary perturbations. Fig. 9 illustrates this range and the present limit from OVRO7.

8 Cosmic Confusion??

The CMB anisotropy power spectrum entails thousands of C_ℓ's and depends only upon a handful of parameters, (n_s, n_t, r, h, Ω_B, Ω_Λ, $\ldots$). One may have hoped that measurements of the power spectrum would be able to test inflation and resolve independently each of the parameters. Unfortunately, it is possible to continuously vary certain combinations of the cosmological parameters without significantly changing the power spectrum. We refer to this "degeneracy" as *cosmic confusion*.[48, 26] It means that CMB anisotropy experiments can localize a hypersurface in cosmic parameter space but the likelihood along that hypersurface will hardly vary. This confounds efforts to separately determine the values of cosmological parameters.

The degree of cosmic confusion depends on the range in ℓ over which the anisotropy spectrum is measured and the precision of the measurement. It seems possible that measurements over the next decade will determine the spectrum from small ℓ through the first Doppler peak $\ell \sim 300$ with an error comparable to cosmic variance. In this section, we will assume that these observations can be made and show that cosmic confusion is a significant problem. If precise observations are restricted to this limited range of multipoles, other cosmic observations must be combined with CMB anisotropy measurements to break the degeneracy in fitting cosmological parameters. The degree of cosmic confusion can be reduced if even more of the power spectrum can be determined precisely through the second Doppler peak ($l \sim 500$ or $sim10$ arcminute scales). However, mapping the full sky at such high resolution is an extraordinary challenge and, even if one succeeds, it is possible that foregrounds will obstruct measurements at these small angular scales. Hence, it seems reasonable, at least for the next decade, to consider the more conservative case in which precise measurements are made only through the

first Doppler peak.

It should be emphasized at the outset that extraordinarily valuable information can be gained from microwave background measurements in spite of cosmic confusion. For example, as discussed under Milestone 2, it is possible to test with high precision whether the large-scale spectrum is scale-free and measure the spectral slope. As shown in Fig. 6, there is negligible interference due to uncertainties in other cosmological parameters. Hence, it is certainly possible to test unambiguously some of the key predictions of inflation.

However, one may have hoped for more. Some may argue that Harrison and Zel'dovich made the case for a scale-invariant spectrum without invoking inflation [8]. In this case, verifying the additional relations Eqs. (21-23), which have no natural explanation from Harrison and Zel'dovich's point-of-view, would be significant, added support for inflation (and there is the obvious converse). Also, it would have been a tremendous boon if other cosmological parameters could be unambiguously determined. This is where cosmic confusion disappoints. All is not lost, though. In some cases (see, for example, the comments at the end of this section), the degree of confusion is minor. Even in the worst cases, powerful relations result when CMB measurements are combined with other astrophysical measurements.

As an illustration of cosmic confusion, consider a baseline (solid line) spectrum ($r = 0|n_s = 1, \mathrm{h} = 0.5, \Omega_\Lambda = 0$). Increasing Ω_Λ (or decreasing h) enhances small-angular scale anisotropy by reducing the red shift $z_{\rm eq}$ at which radiation-matter equality occurs. CMB anisotropy experiments can determine either $r|n_s$, Ω_Λ, or h quite accurately if the other two parameters are known [26]. However, cosmic confusion arises if $r|n_s$, Ω_Λ and h vary simultaneously. Fig. 10 shows spectra for models lying in a two-dimensional surface of $(r|n_s, \mathrm{h}, \Omega_\Lambda)$ which produce nearly identical spectra. In one case, $r|n_s$ is fixed, and increasing Ω_Λ is nearly compensated by increasing h. In the second case, h is fixed, but increasing Ω_Λ is nearly compensated by decreasing n_s.

Further cosmic confusion arises if we consider ionization history. Let z_R be the red shift at which we suppose sudden, total reionization of the intergalactic medium. Fig. 11 compares spectra with standard recombination (SR), no recombination (NR) and late reionization (LR) at $z_R = 50$, where $h = 0.5$ and $\Omega_\Lambda = 0$. Reionization or no recombination suppresses the small angular scale anisotropy, which can be confused with a decrease in n_s (see figure). Inflation-inspired models, *e.g.*, cold dark matter models, are likely

Figure 10: Examples of different cosmologies with nearly identical power spectra and band powers. The solid curve is $(r = 0|n_s = 1, h = 0.5, \Omega_\Lambda = 0)$. The other two curves explore degeneracies in the $(r = 0|n_s = 1, \mathrm{h}, \Omega_\Lambda)$ and $(r|n_s, h = 0.5, \Omega_\Lambda)$ planes. In the dashed curve, increasing Ω_Λ is almost exactly compensated by increasing h. For the dot-dashed curve, the effect of changing to $r = 0.42|n_s = 0.94$ is nearly compensated by increasing Ω_Λ to 0.6. Restricting attention to the range from $\ell = 2$ through the first Doppler peak, one finds that the curves are difficult to distinguish, especially given cosmic variance (vertical hashing is one-sigma variance). If precision measurements of the second Doppler peak or beyond can be made, it should be possible to distinguish some of the models.

Figure 11: Power spectra for models with standard recombination (SR), no recombination (NR), and 'late' reionization (LR) at $z = 50$. In all models, $h = 0.5$ and $\Omega_\Lambda = 0$. NR or reionization at $z \geq 150$ results in substantial suppression at $\ell \geq 100$. Models with reionization at $20 \leq z \leq 150$ give moderate suppression that can mimic decreasing n_s or increasing h; for example, compare the $n_s = 0.95$ spectrum with SR (black, dashed) to the $n_s = 1$ spectrum with reionization at $z = 50$ (grey solid).

to have negligibly small z_R.

The results can be epitomized by some simple rules-of-thumb: Over the $30' - 2^\circ$ range, the C_ℓ's for fixed ℓ are roughly proportional to the maximum of C_ℓ at the top of the first Doppler peak. The maximum (corresponding to $\sim .5^\circ$ scales) is exponentially sensitive to n_s. Since scalar fluctuations account for the Doppler peak, the maximum decreases as the fraction of tensor fluctuations (or r) increases. The maximum is also sensitive to the red shift at matter-radiation equality (or, equivalently, $(1-\Omega_\Lambda)h^2$), and to the optical depth at last scattering for late-reionization models, $\sim z_R^{3/2}$. These observations are the basis of an empirical formula (accurate to $\lesssim 15\%$)

$$\frac{C_\ell}{\langle C_\ell \rangle_{dmr}}\Big|_{max} \approx A\, e^{B\,\tilde{n}_s}\ , \tag{27}$$

with $A = 0.15$, $B = 3.56$, and

$$\begin{aligned} \tilde{n}_s &\approx n_s - 0.28 \log(1 + 0.8r) \\ &\quad -0.52[(1 - \Omega_\Lambda)h^2]^{\frac{1}{2}} - 0.00036\, z_R^{3/2} + .26 \ , \end{aligned} \tag{28}$$

where r and n_s are related by Eq. (23) for generic inflation models. ($\tilde{n}_s$ has been defined such that $\tilde{n}_s = n_s$ for $r = 0$, $h = 0.5$, $\Omega_\Lambda = 0$, and $z_R = 0$.) Hence, the predicted anisotropy for any experiment in the range $10'$ and larger is not separately dependent on n_s, r, Ω_Λ, etc.; rather, it is function of the combination $\tilde{n}_s$.

Eq. (27) implies that the CMB anisotropy measurements are exponentially sensitive to $\tilde{n}_s$. Hence, we envisage that $\tilde{n}_s$ will be accurately determined in the foreseeable future. Then, Eq. (28) implies that the values of the cosmological parameters are constrained to a surface in parameter-space. Cosmological models corresponding to any point on this surface yield indistinguishable CMB anisotropy power spectra (thru the first Doppler peak). To determine which point on the surface corresponds to our Universe requires other astrophysical measurements. For example, limits on the age of the Universe from globular clusters, on h from Tully-Fisher techniques, on n_s from galaxy and cluster counts, and on Λ from gravitational lenses all reduce the range of viable parameter space to a considerable degree. It is by this combination of measurements that the CMB power spectrum can develop into a high precision test of cosmological models [21, 49].

The difficulty posed by cosmic confusion depends on which way the experimental results break. We have already stated that cosmic confusion can be lifted if precise measurements can be made through the second Doppler peak and beyond. But it should also be noted that, even if we are limited to the first Doppler peak only, there are situations in which cosmic confusion is significantly reduced. Consider our baseline spectrum, $(r = 0|n_s = 1, \mathrm{h} = 0.5, \Omega_\Lambda = 0)$. If measurements indicate a first Doppler peak which is significantly below the baseline value, then there are numerous effects which might be responsible: tilt ($n_s < 1$), increased gravitational wave contribution (r), increased Hubble parameter ($h > 0.5$), decreased $\Omega_B < 0.05$, reionization, or perhaps cosmic defects. On the other hand, if measurements indicate a first Doppler peak that is significantly higher than the baseline value, the causative effects might be: upward tilt ($n_s > 1$), cosmological constant $\Omega_\Lambda > 0$, decreased Hubble parameter ($h < 0.5$), or

Figure 12: The percentage polarization in $\Delta T/T$ versus multipole moment ℓ predicted for an inflationary model with $n_s = 0.85$, $h = 0.5$, cold dark matter, and standard recombination. (For this value of n_s, inflation predicts equal scalar and tensor contributions to the unpolarized quadrupole.) The upper panel represents the prediction for standard recombination and the lower panel is for a model with no recombination.

increased $\Omega_B > 0.05$. Of these four possibilities, the first three are extremely unattractive due to theoretical and observational constraints; the data strongly suggest $\Omega_B > 0.05$ with all other parameters held at their original values. The degree of real cosmic confusion is considerably less in the second case. Since several of the present experiments (including MAX) suggest anisotropy somewhat greater than the baseline value, these comments are especially timely.

9 Polarization

In the discussion thus far, we have focused on what can be learned from the CMB anisotropy measurements based on the power spectrum only. The power spectrum represents only the two-point temperature correlation function. From a CMB anisotropy map, one can hope to measure three- and higher-point correlation functions, for example, to test for non-gaussianity of the primordial spectrum. Another conceivable test is the CMB polarization. Measurements of the polarized temperature autocorrelation function [50] and of the cross-correlation between polarized and unpolarized anisotropy [51] are further tests of inflation.

Calculations for realistic parameters, though, suggest that the polarization is unlikely to be detected or to provide particularly useful tests of cosmological parameters [52]. For example, it had been hoped that large-scale (small ℓ) polarization measurements would be useful for discriminating scalar and tensor contributions to the CMB anisotropy, thereby measuring r. In the upper panel of Fig. 12, we show the percentage polarization (in $\Delta T/T$) for scalar and tensor modes for a model with $r = 1$ and $n_s = .85$, an example where there are equal tensor and scalar contributions to the quadrupole moment. The figure shows that, indeed, there is a dramatically different

polarization expected for scalar versus tensor modes for small ℓ. However, the magnitude of the polarization is less than 0.1%, probably too small to be detected in the foreseeable future. On scales less than one degree ($\ell > 100$), the total polarization rises and approaches 10%, a more plausible target for detection. However, the tensor contribution on these angular scales is negligible, so detection does not permit us to distinguish tensor and scalar modes. In fact, the predictions are relatively insensitive to the cosmological model, a notable exception being the reionization history. The lower panel of Fig. 12 illustrates the prediction for a model with no recombination. The overall level of polarization is increased. The tensor contribution is suppressed relative to scalar, so polarization remains a poor method of measuring r. However, the polarization at angular scales of a few degrees ($\ell \approx 50$) rises to nearly 5%, perhaps sufficient for detection. An observation of polarization at these angular scales would be consistent with a non-standard reionization history.

Not only is the predicted polarization small in all cases, but little is known about what the foreground contamination will be. At a minimum, the foreground from dust, synchrotron, etc., interferes with measurements at the 1% level [53]. Polarization measurements can provide useful corroborating evidence or surprises. For example, a sizable polarization (20%, say) would be unexpected in all models. However, if inflation is correct, then (unpolarized) anisotropy, rather than polarization, appears to be the most useful discriminant for the foreseeable future.

10 Conclusions

The next decade is sure to be a historic period in the endeavor to understand the origin and evolution of the Universe. In this article, we have focused only on measurements of the cosmic microwave background. We have shown how these measurements alone will allow us to test the inflationary hypothesis and place new constraints on almost all cosmological parameters. Table 2 below summarizes the specific sequence of five milestones in CMB anisotropy experiments that need to be achieved to accomplish these goals, the range of multipole moments (C_ℓ) that need to be probed, and which aspects of inflationary predictions is tested at each milestone. Passing all milestones is overwhelming support for inflation; failing to pass one or more milestones is either invalidation or, at least, indication of some significant, additional

surprise.

The ability to separately resolve inflationary and other cosmological parameters from microwave background measurements alone is limited by cosmic confusion, the phenomenon that different choices of cosmological parameters result in virtually identical CMB anisotropy spectra. Confusion is not a significant problem if the observed anisotropy at the first Doppler peak ($\sim 0.5^\circ$) is found to be large or if precise measurements can be extended to the second Doppler peak and beyond (see remarks at the end of Section 8). However, it is not clear whether either of these conditions will be met. Even in that case, it is still possible to use the CMB anisotropy measurements as a powerful tool for determining cosmological parameters, but only after combining CMB results with other known astrophysical constraints.

Another extraordinary series of efforts during this same decade entails measurements of large-scale distribution and velocities of galaxies. These observations probe the distribution of matter on scales generally smaller than but overlapping the range probed by microwave background experiment. The ultimate goal is to find a simple, intuitive theory which joins together the microwave background and large-scale structure observations.

The present situation is summarized in Fig. 13, in which the predictions of dark matter models (in linear approximation only) are superimposed [54]. (Non-linear corrections have been computed for some models via numerical simulation.) At present, the cold dark matter models are the simplest from a theoretical point-of-view, but predict too much power on small scales when normalized to COBE DMR. Adding new dark matter components, as in mixed dark matter models, leads to a better fit, but at the cost of introducing the special condition of nearly equal hot and cold matter densities that is difficult to explain from microphysics. Tilting the spectrum to reduce power at small scales is naturally incorporated into inflation, but the numerical simulations do not fit large-scale structure as successfully. An interesting puzzle appears to be brewing.

Microwave background and large-scale structure observations are dramatic examples of the transformation of cosmology from metaphysics to hard science. These, combined with the more classical cosmological measurements of h, Ω, and Λ, are rapidly evolving into highly precise tools for testing theories and measuring fundamental parameters. As the new millennium commences, cosmological science stands at a critical crossroads: we do not know whether the Universe can be explained solely on the basis of physi-

Figure 13: Comparison of matter density power spectra for cold dark matter (CDM) tilted cold dark matter (TCDM), hot dark matter (HDM) and mixed cold and hot dark matter models (MDM) of largescale structure formation. All theoretical curves are normalized to COBE DMR and include only linear approximation. (Non-linear corrections become important at small angular scales (≤ 10 Mpc).

cal laws that can be tested locally, or whether knowledge of initial conditions or new physics beyond our grasp is required. Evidence that the Universe is scale-free (*e.g.*, spatially flat, scale-free primordial spectrum of fluctuations) is consistent with the notion that the physical laws governing the Universe involve microphysical scales only, as suggested by present understanding. Evidence that the Universe is measurably open (or closed) or that there are large-scale features in the primordial spectrum suggests initial conditions or coincidences or new physical laws that may not be probed by other means. The series of measurements anticipated in the next decade may determine the future path of cosmology, and, thereby, the ultimate limitations on human comprehension of the Universe.

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Appendix: Flat Band Power Estimates

The observational results reported in Figs. 4, 5, 8 and 9 represent "flat band power estimates" derived for each experiment. In this appendix, we explain the reasoning behind band power estimation; we briefly discuss the particular cases of GACF (gaussian auto-correlation function) and flat band power estimation, and approximate methods for converting from one estimate to another.

Each experiment, depending on the geometry, scanning strategy and detectors, has different sensitivity to the C_ℓ's which can be expressed through a filter function [43], W_ℓ. A number which can be directly extracted from experiment is the rms variation in temperature,

$$(\Delta T/T)^2_{rms} \equiv \sum_\ell \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} \left[\frac{1}{2\pi}\ell(\ell+1)C_\ell^{sky}\right] W_\ell; \tag{29}$$

the square brackets enclose the expression for the power spectrum of the real sky described multipole moments C_ℓ^{sky}. It is difficult to compare $(\Delta T/T)_{rms}$ directly with theoretical predictions for three reasons: (1) the value of $(\Delta T/T)_{rms}$ is dependent upon the normalization and shape of the filter function, W_ℓ; (2) a significant fraction of $(\Delta T/T)_{rms}$ is noise; and, (3) the value of $(\Delta T/T)_{rms}$ itself gives no information about the functional form of the real sky power spectrum (that is, how $\ell(\ell+1)C_\ell^{sky}$ varies with ℓ).

A first improvement is to compute the rms band power, defined by

$$\frac{1}{2\pi}\bar{\ell}(\bar{\ell}+1)C_{\bar{\ell}} \equiv \frac{(\Delta T/T)^2_{rms}}{\sum_\ell \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} W_\ell}, \tag{30}$$

where

$$\bar{\ell} \equiv \frac{\sum_\ell \frac{\ell+\frac{1}{2}}{\ell+1} W_\ell}{\sum_\ell \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} W_\ell} \tag{31}$$

is the mean value of ℓ over the filter function band. (Some authors also delineate span of ℓ corresponding to the half-width of the filter function by a horizontal error bar;[44, 45] we have not do so in the interests of clarity.) Band power is independent of the normalization of W_ℓ, and only weakly dependent on its shape. However, problems (2) and (3) remain.

A second step is to introduce "$\langle$*functionalform*$\rangle$ band power estimation," a fit to the data assuming some particular functional form for the power

spectrum. The assumed functional form could be the prediction of some specific theoretical model, such as the spectra shown in Fig. 2; however, there are too many weakly constrained parameters in these models at present. Hence, a less biased approach for now is to assume a simpler form, such as gaussian (GACF) or flat, which depends on only a few free parameters. A maximum likelihood fit of the data to the functional form fixes the best-fit choices of the parameters. The $\langle functionalform \rangle$ band power estimate is then:

$$\frac{1}{2\pi}\bar{\ell}(\bar{\ell}+1)C_{\bar{\ell}} \equiv \sum_{\ell} \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} \left[\frac{1}{2\pi}\ell(\ell+1)C_{\ell}^{bf}\right] W_{\ell}, \tag{32}$$

where $\ell(\ell+1)C_{\ell}^{bf}/2\pi$ has the assumed functional form with free parameters fixed at the best-fit values. Best-fitting the data to a functional form which resembles the real sky has the advantage that it filters out noise. By including parameters which change the shape of the assumed power spectrum, some shape information can be extracted.

A "flat band power estimate" assumes $\ell(\ell+1)C_{\ell}^{bf}/2\pi$ is ℓ-independent; the one free parameter is the amplitude. A gaussian autocorrelation function (GACF) estimate assumes that the correlation function is gaussian:

$$\begin{aligned} C(\theta) &= \left\langle \tfrac{\Delta T}{T}(\mathbf{x})\tfrac{\Delta T}{T}(\mathbf{x}') \right\rangle \\ &= C(0)\,\tfrac{\theta_c^2}{\theta_c^2+2\sigma^2}\exp\{-\theta^2/[2(\theta_c^2+2\sigma^2)\} \end{aligned}$$

where σ is the FWHM beamwidth of the experiment divided by $\sqrt{8\ln 2}$. This corresponds to a gaussian power spectrum $\ell(\ell+1)C_{\ell}^{bf}/2\pi$. The spectrum has two free parameters, the correlation angle θ_c and the best-fit amplitude $C(0)$ for the given θ_c. (Some experiments report two-parameter best-fits. Others report single-parameter fits to $C(0)$ for fixed θ_c.)

Ultimately, a likelihood fit over a spectrum of theoretical models is the proper way to determine the best-fit model. However, it is difficult to show widely disparate models and many different parameters within a single likelihood plot. It would be highly desirable if one could somehow find unbiased band power estimates extracted from the experiment that can be compared directly to the theoretical power spectra.

After testing various forms, Bond *et al.* [44, 45] have demonstrated that flat band power estimates are a good choice for direct comparison with theory, and this is what has been applied in Figs. 4, 5, 8 and 9. If the real sky

power spectrum is similar to one of the theoretical curves, the flat band power approximation is clearly a better than a gaussian (GACF) one for the plateau region (shown in Fig. 4); flat band power estimatation works surprisingly well for the large ℓ's as well. For example, if one computes flat band power estimates assuming the theoretical predictions, the estimates closely hug the theoretical curves even through the Doppler peaks [45].

As described above, the proper method of obtaining the flat band power estimate is through a full, likelihood fit to the data assuming the flat spectrum. At this point, though, many experimental groups only publish a GACF analysis. The following five easy steps are an approximate conversion formula works well for most experiments. From a reported θ_c and $\Delta T/T_{GACF}$:

1. Compute $\ell_c \equiv 1/[2\sin(\theta_c/2)]$.

2. Compute the filter function W_ℓ for the given experiment.

3. Compute
$$N \equiv \sum_\ell \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} \left(\frac{\ell+\frac{1}{2}}{\ell_c+\frac{1}{2}}\right)^2 \exp\left[-\frac{1}{2}\left(\frac{\ell+\frac{1}{2}}{\ell_c+\frac{1}{2}}\right)^2\right] W_\ell$$

4. Compute
$$D = \sum_\ell \frac{\ell+\frac{1}{2}}{\ell(\ell+1)} W_\ell$$

5. Then the flat band power estimate for $\ell(\ell+1)C_\ell$ is
$$\frac{\bar{\ell}(\bar{\ell}+1)C_{\bar{\ell}}}{2\pi} = \left(\frac{\Delta T}{T}\right)^2_{GACF} \frac{N}{D}$$

.

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Milestone	Range of ℓ	What It Tests
1. Large Scale Fluctuations	$2 \lesssim \ell \lesssim 30$	Spectral Amplitude
2. Plateau at Intermediate Scales	$10 \lesssim \ell \lesssim 100$	Spectral shape/slope
3. First Doppler Peak	$100 \lesssim \ell \lesssim 300$	
a. Value of ℓ at the maximum		Flatness
b. Height		Constraints on h, Ω_B, Ω_Λ and reionization?
4. Second & Higher Doppler Peaks	$300 \lesssim \ell \lesssim 800$	Constraints on $\Omega_B h^2$, CDM vs. MDM
5. Damping	$\ell \gtrsim 1000$	Silk Effect, Cosmo. parameters?

Table 2: Five Milestone Tests of Inflation and Dark Matter Models of Large-scale Structure